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Integrability test for spin chains

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Abstract. We examine a simple heuristic test of integrability for quantum chains. This test is applied to a variety of systems, including isotropic spin-1 models with nearest-neighbour interaction, Potts-type, inhomogeneous spin-$\frac{1}{2}$ chains, and a multiparameter family of spin-$\frac{1}{2}$ models generalizing the *XYZ* chain, with next-to-nearest neighbour interactions and bond alternation. Within the latter family we determine all the integrable models with an $o(2)$ symmetry.

1. Introduction

For Hamiltonian systems, the common definition of quantum integrability mimics the Liouville–Arnold definition of classical integrable Hamiltonian systems: a quantum system with N degrees of freedom is called integrable, if it possesses N non-trivial, functionally independent and mutually commuting conservation laws.

For classical continuous systems, the proof of integrability usually amounts to displaying a Lax or zero-curvature formulation. Although this is not always easy, there are various other manifestations of integrability that can be probed, such as a bi-Hamiltonian formulation, non-trivial symmetries, prolongation structures or higher-order conserved charges (see e.g. [1]). Furthermore, one can apply a systematical integrability test, based on the Painlevé property [2]. For classical discrete systems, there are related integrability indicators [3]. But except for higher-order conservation laws, these integrability signals are no longer available for quantum chains.

On the other hand, the integrability of quantum chains is usually demonstrated rather indirectly, by showing that the model can be solved by the coordinate Bethe ansatz or that the Hamiltonian can be derived from a commuting family of transfer matrices related to the Yang–Baxter equation. But these are only sufficient conditions for integrability. Moreover, testing these sufficient conditions is often not easy‡.

It is thus clearly desirable to design a more general, simple and efficient integrability test for quantum chains. Here we propose a simple test based on the existence of a local non-trivial three-point charge H_3, the higher-order conservation law just above H_2, the defining Hamiltonian of the model. Locality means that the interaction involving a certain set of sites disappears when the distances between them become sufficiently large. We will indicate below why for quantum chains, one can expect that the existence of H_3 should be a necessary and in some cases a sufficient condition for integrability. From the computational

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‡ Note, however, that the Yang–Baxter equation implies a relatively simple equation, known as the Reshetikhin condition. See section 2 for a discussion of this point.

point of view, the advantage of this simple-minded approach is that the number of possible candidates for the three-point charge is usually not exceedingly large, and they are often restricted by the symmetries of the system.

We stress that the proposed test is heuristic. Its applicability in a large number of situations leads us to present it in the form of two conjectures. These conjectures, in turn, were motivated by observations that have not been proved in general. Thus, few rigorous results are displayed. Our aim is to try to pin down a simple integrability indicator. The application of the test to more examples will reveal its genuine value and/or its limitations. But the search for a simple integrability test is important. When modelling a physical situation in terms of a multi-parameter Hamiltonian, it is of great interest to know whether the model is integrable for some values of the parameters. Even though the integrable point may not correspond to the most relevant physical situations, it provides a convenient starting point for a perturbative treatment of non-integrable models of interest.

2. Conjectured integrability tests for quantum chains

2.1. *The class of models to be considered*

The test is formulated for the class of translationally invariant models with nearest-neighbour interactions. These models are defined on a lattice Λ which is either finite with periodic boundary conditions (i.e. $\Lambda = \{1, \ldots, N\}$, with $N + 1 \equiv 1$) or infinite (i.e. $\Lambda = Z$).

Let $\{S_i^a\}$ denote a set of quantum operators (distinguished by the index a), acting non-trivially only in a Hilbert space V_i. (We will assume in this work that all the V_i's are finite-dimensional; in general they don't have to be identical.) The full space of states of such chain is then the tensor product $\bigotimes_{j\in\Lambda} V_j$. The class of Hamiltonians under consideration has then the following form:

$$H_2 = \sum_{j\in\Lambda}[g_j + h_{j,j+1}] \tag{2.1}$$

where $g_j = g(S_j)$ describes interactions at site j and $h_{j,j+1}$ is some site-independent function of S_j, S_{j+1} which describes nearest-neighbour interactions.

We stress that the restriction to nearest-neighbour interactions is not as severe as it might appear at first sight, since this class of models actually contains, when appropriately reformulated, any model involving binary interactions with finite range. Indeed, any model with a finite interaction range k_0 (where $k_0 = 2$ corresponds to nearest-neighbour interactions) can be equivalently described in terms of nearest-neighbour interactions just by grouping together k_0 consecutive sites into a single site on which a vectorial spin-like variable would live. Similarly, any model with a more general invariance under a shift $j \to j + j_0$ can be made invariant with respect to a translation by a single unit, by grouping together j_0 consecutive sites.

2.2. *Preliminary observations*

(i) H_3 *cannot be absent due to a symmetry or a null-vector.* Since the early days of soliton theory, it has been clear that the existence of a non-trivial conservation law (beyond those associated with the standard conservation of mass or charge, momentum and energy) is a very strong indication of the existence of an infinite number of additional non-trivial conservation laws. But for continuous systems, either classical or quantum, one cannot focus the attention on the existence or non-existence of the conservation law with a degree just above the one which usually plays the role of the Hamiltonian. The reason is simply that symmetry considerations can prevent the existence of H_n for a set of values of n. Take for instance the KdV equation

$$u_t + u_{xxx} + 6uu_x = 0 \tag{2.2}$$

for which u has degree 2 in the normalization where ∂_x has degree 1. Due to a hidden Z_2 invariance, there are no conservation laws with densities of odd degree. More generally, for Toda systems related to Lie algebras, the values of n for which H_n is non-zero are related to the exponents of the corresponding Lie algebra [4]. But notice that when the conservation laws are calculated by a recursive method, the conserved densities associated with charges that vanish by symmetry are not found to be zero, rather they are total derivatives.

As indicated above, these considerations apply for both classical and quantum systems. But in the quantum case, there is another possible source for the absence of a conserved charge of given order, which is that the density can be exactly proportional to a null vector. Again this is observed in the quantum KdV case, where for special values of the central charge, the free parameter in the defining commutation relation, some conserved densities become proportional to null vectors of the Virasoro algebra [5].

Now, our point here is that both these possibilities are absent in the case of quantum chains. In particular, it is clear that on the lattice there is no room for total derivatives. Furthermore, null vectors appear not to be relevant if the spaces V_i are finite dimensional†.

(ii) *Periodic spin chains whose integrability is rooted in the Yang–Baxter equation have a non-zero* H_3. On the other hand, for systems whose integrability can be traced back to commuting transfer matrices, the conserved charges are obtained from the expansion of the logarithm of the transfer matrix [6] and H_3 is never absent. This can be easily seen for fundamental systems [7], characterized by a Lax operator $L_n(\lambda)$ proportional to the model's R matrix. The result actually follows directly from the boost construction described in sections 2.5 and 2.6.

The existence of a non-trivial third-order charge can also be proved for a more general class of integrable models, characterized by the condition $L_n(0) = P_{n0}$, where P_{n0} denotes the permutation operator and the index zero refers to the quantum space V_0 on which the matrix entries of L_n act‡. The second logarithmic derivative of the transfer matrix

$$T(\lambda) = \mathrm{Tr}_{V_0} L_N(\lambda) \ldots L_1(\lambda) \tag{2.3}$$

becomes

$$H_3 \sim \sum_{j\in\Lambda} [h_{j,j+1}, h_{j+1,j+2}] - \sum_{j\in\Lambda} h_{j,j+1} h_{j,j+1} + \sum_{j\in\Lambda} T^{-1}(0) \mathrm{Tr}_{V_0} \left(P_{N0} \ldots \frac{\mathrm{d}^2 L_j}{\mathrm{d}\lambda^2}(0) \ldots P_{10} \right). \tag{2.4}$$

The last two terms involve only nearest-neighbour interactions, while the first is a sum over triples of consecutive spins. If one excludes a pathological case in which all adjacent links commute, the three-point part of the charge defined above is non-trivial§. Note that the other two terms are absent for fundamental models.

† Constraints imposed on the factor spaces V_i (i.e. realized via a projection onto some subspace of V_i) may 'project out' some charges of an integrable model—in other words some of the conserved charges evaluated in a restricted space of states may conceivably vanish. Such constraints would thus have an effect similar to null vectors. However, situations like this can be avoided by considering the conservation laws in the full (unrestricted) Hilbert space.

‡ Note that such an L matrix is not necessarily itself a solution of the Yang–Baxter equation. An example of such a transfer matrix, but which does not define a fundamental model, is provided by the transfer matrix of the Hubbard model found by Shastry [8].

§ For example, such a pathological situation arises for the 'chopped XXZ' model introduced in section 3.

2.3. *A conjectured necessary condition for integrability of a quantum chain*

The above considerations motivate the following conjecture for the class of systems described by (2.1).

Conjecture 1. A translationally invariant periodic quantum spin chain, with a Hamiltonian H_2 involving at most nearest-neighbour interactions, is integrable only if there exists a non-vanishing local independent charge H_3, which is a sum of terms coupling the spins of, at most, three sites, which commutes with H_2 for all chain sizes $N \geqslant 3$.

This conjecture implies then a simple test consisting in establishing the existence of such H_3†. In particular, one may conclude that a spin chain is non-integrable, by demonstrating the non-existence of a non-trivial H_3. On the other hand, if such a charge exists, the system is likely to be integrable; but of course its integrability has to be proved independently. Actually, it also appears that for a large class of systems (and, in particular, for the models considered here), the mere existence of H_3 is enough to guarantee the existence of an infinite family of conserved charges in involution.

Notice that there is a general class of quantum chains for which the existence of H_3 automatically ensures the existence of an infinite number of commuting charges. These are the self-dual systems satisfying the Dolan–Grady condition [10]. It includes, in particular, the Z_n generalization of the Ising model related to the two-dimensional Potts model. The formulation of this condition is reviewed in appendix A.

2.4. *Clarifying comments related to the formulation of conjecture 1*

Some elements entering in the formulation of the conjecture deserve clarification.

(i) *Independence of the charges.* The charge H_3 in the above conjecture should be independent of the Hamiltonian and of possible charges of lower order (such as, e.g. the components of the total spin). For general quantum integrable models, the issue of functional independence can be quite complicated; it is indeed a major difficulty in formulating a general definition of quantum integrability in a completely rigorous way‡. However, functional independence of the charges can usually be easily verified for spin chains with short-range interactions: the leading term of H_n contains n adjacent interacting spins; the leading terms of H_n and H_m for $m \neq n$ being clearly distinct, the corresponding charges are linearly independent. Furthermore, such a cluster of n adjacent spins cannot be obtained from a product of lower-order charges (since that would also generate terms with n non-adjacent spins).

(ii) *Stability of the charge under a variation of the chain length.* The last item in the above conjecture ensures that the existence of H_3 should not be affected by a change $N \to N+k$, where k is an arbitrary positive integer. This stability requirement can be simply illustrated

† A related integrability test has been studied in [9]. However, these authors confined their test to the search of one non-trivial conservation law (not necessarily H_3) in spin-$\frac{1}{2}$ models, with the simultaneous existence of a ladder operator providing a recursive scheme for the calculation of the other conservation laws. This is certainly less general and less constructive than the test proposed here.

‡ Dropping the requirement of functional independence does not lead to a meaningful definition of integrability [11]. In the context of spin chains this can be easily seen. Consider an isotropic chain, for which the Hamiltonian commutes with all the components of the total spin. Arbitrary powers of any of the spin components yield then a set of mutually commuting conserved charges. Removing the requirement of functional independence in the definition of integrability would therefore render any isotropic spin chain automatically integrable.

for the *XXX* model defined by the Hamiltonian

$$H = \sum_{i=1}^{N} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_{i+1} \tag{2.5}$$

(where as usual periodic boundary conditions are assumed). This Hamiltonian commutes with any component of the total spin

$$\mathcal{S}^a = \sum_{i=1}^{N} \sigma_i^a \,. \tag{2.6}$$

A simple calculation shows that for $N = 4$ the quantity

$$H' \equiv \sum_{i=1}^{N} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_{i+2} \tag{2.7}$$

is conserved. H' is also conserved for $N = 5$ but not for any higher value of N. The reason for this behaviour is that for $N = 4$ and 5, H' can be regarded as a non-local charge. Such non-local charges can typically be obtained from powers of the local charges. More exactly, for $N = 4$ we have the following identity (modulo an additive constant):

$$\mathcal{S}^a \mathcal{S}^a = 2H + H' \,. \tag{2.8}$$

Similarly, for $N = 5$

$$\mathcal{S}^a \mathcal{S}^a = 2H + 2H' \,. \tag{2.9}$$

However, for $N > 5$, H' is not related to $\mathcal{S}^a \mathcal{S}^a$ or other non-local charges and is no longer conserved. For $N = 6$ for instance, we have instead

$$\mathcal{S}^a \mathcal{S}^a = 2H + 2H'' \tag{2.10}$$

where

$$H'' \equiv \sum_{i=1}^{N} [\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_{i+2} + \tfrac{1}{2} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_{i+3}] \tag{2.11}$$

contains contributions with one and two holes. Thus the conservation of H' is not preserved under the change $N \to N + 2$; the same is true for H''. This is actually typical for all such 'accidental' charges, whose conservation for some particular values of N is due to an accidental identity (relating them to non-local charges), which is true only for these particular values of N.

(iii) *Locality versus non-locality.* The above example illustrates another issue: it is not always easy to distinguish the non-local charges (e.g. H') from the local charges (e.g. H) for finite chains. For an infinite chain, local and non-local expressions are quite distinct (the latter contain interactions between arbitrarily distant spins). On the other hand, for the *XXX* model with $N = 4, 5$, one hole in the expression of a 2-spin conserved law reflects 'non-locality'! However, as exemplified above, the form of those non-local charges that can be written as powers of local charges is strongly N-dependent; this property makes them easily detectable. Notice also that the local charges for a finite chain may be defined non-ambiguously from the densities of the first N charges of the infinite chain.

2.5. A conjectured sufficient condition for integrability based on the boost operator

The integrable points in a multi-parameter space of general spin-chain Hamiltonians can usually be simply characterized by the occurrence of a dynamical symmetry. Related to such symmetry is the existence of a ladder operator B, acting on the conservation laws as

$$[B, H_n] = H_{n+1} \tag{2.12}$$

where H_n denotes a charge with at most n adjacent interacting spins†. This motivates the formulation of a simple conjectured sufficient condition for integrability, based on the existence of a ladder operator B, in conjunction with the presence of a non-trivial charge H_3.

Conjecture 2. A translationally invariant periodic quantum chain, with a Hamiltonian H_2 involving at most nearest-neighbour interactions, is integrable if there exists an operator B such that for all chains of length $N \geqslant 3$, $[B, H_2]$ is non-trivial and

$$[[B, H_2], H_2] = 0\,. \tag{2.13}$$

In contrast to the first conjecture, the approach here is more constructive: it indicates how H_3 can be built, i.e. as $[B, H_2]$. This constructive aspect presupposes that we can easily guess the form of B. In all the cases we have considered, such a B turns out to be proportional to the first moment of an appropriately symmetrized form of the Hamiltonian H_2, i.e. with

$$H_2 = \sum_{j\in\Lambda} h_{j,j+1} \qquad \text{for} \quad h_{j,j+1} \text{ symmetric in } j,\ j+1\,. \tag{2.14}$$

B is found to be

$$B = \sum_{j\in\Lambda} j\ h_{j,j+1}\,. \tag{2.15}$$

The condition of commutativity of H_2 and H_3 assumes then a particularly simple form:

$$\sum_{j\in\Lambda} [h_{j,j+1} + h_{j+1,j+2}, [h_{j,j+1}, h_{j+1,j+2}]] = 0\,. \tag{2.16}$$

The derivation of this result is very simple. Starting from

$$\begin{aligned} H_3 = [B, H_2] &= \sum_{j\in\Lambda} (j-1)[h_{j-1,j}, h_{j,j+1}] + (j+1)[h_{j+1,j+2}, h_{j,j+1}] \\ &= -\sum_{j\in\Lambda} [h_{j,j+1}, h_{j+1,j+2}] \end{aligned} \tag{2.17}$$

one then enforces

$$\begin{aligned} [H_3, H_2] = \sum_j &[[h_{j-2,j-1}, h_{j-1,j}], h_{j,j+1}] + [[h_{j-1,j}, h_{j,j+1}], h_{j,j+1}] \\ &+ [[h_{j,j+1}, h_{j+1,j+2}], h_{j,j+1}] + [[h_{j+1,j+2}, h_{j+2,j+3}], h_{j,j+1}] = 0\,. \end{aligned} \tag{2.18}$$

Using the Jacobi identity, the first term on the right-hand side can be written in the form $-[h_{j-1,j}, h_{j,j+1}], h_{j,j+1}]$; with the shift $j \to j+2$, it exactly cancels the fourth term. Then, by shifting j by one unit in the second term, we recover (2.16).

From the Jacobi identity, this condition automatically ensures the existence of a second non-trivial conservation law H_4 which commutes with H_2:

$$[H_4, H_2] = [[B, H_3], H_2] = -[[H_3, H_2], B] - [[H_2, B], H_3] = 0\,. \tag{2.19}$$

† Note that we allow for the possibility of a linear combination of lower-order charges $H_{m\leqslant n}$ on the right-hand side of (2.12).

However, showing that $[H_4, H_3] = 0$ and that higher charges commute with H_2 requires additional information. For example, if it is known beforehand that the commutant of H_2 is Abelian, equation (2.16) actually implies the existence of an infinite tower of charges in involution.

2.6. Relation between conjecture 2 and the Reshetikhin condition

It is known (see [12]) that the existence of a ladder operator is a direct consequence of the Yang–Baxter equation for nearest-neighbour interacting chains for which the transfer matrix is a product of R matrices (the so-called fundamental spin chains). But the conjecture is *a priori* independent of the Yang–Baxter equation and in principle there could exists models satisfying (2.16) and not the Yang–Baxter equation. Furthermore, equation (2.16) is easier to test that the Yang–Baxter equation for the related R-matrix.

Actually, for fundamental spin systems, (2.16) can be viewed as a condition for the matrix

$$R(\lambda) = P[I + \lambda H_2 + \mathcal{O}(\lambda^2)] \tag{2.20}$$

to be a solution of the Yang–Baxter equation. This approach leads to the condition

$$[h_{j,j+1} + h_{j+1,j+2}, [h_{j,j+1}, h_{j+1,j+2}]] = X_{j,j+1} - X_{j+1,j+2} \tag{2.21}$$

for some quantity X. This relation first appeared in [13] (see equation (3.20)) and it is attributed to Reshetikhin. An explicit derivation can be found in [14]. The Reshetikhin condition (2.21) is nothing but the local version of (2.16). It appears to be an anticipation of the boost construction of conservation laws. It is also pointed out in [13] that this equation is not satisfied by all integrable systems (which is by now understood from the fact that not all such systems are fundamental); in particular this is the case for the Hubbard model (in agreement with the conclusion in [15] concerning the non-existence of a boost operator).

In the rest of this work we examine the existence of a third-order charge for a number of models. Some of the calculations were performed using MATHEMATICA.

3. Example 1: spin-$\frac{1}{2}$ chains with next-to-nearest neighbour interactions and bond alternation

3.1. Definition of the model

We consider a 10-parameter family of spin-$\frac{1}{2}$ models, which contain, in addition to XYZ-type interactions, next-to-nearest neighbour interactions, bond alternation terms and a magnetic field coupling term:

$$H_2 = \sum_{j\in\Lambda}\left\{\sum_{a=x,y,z}[\lambda_a\sigma_j^a\sigma_{j+1}^a + \lambda'_a(-1)^j\sigma_j^a\sigma_{j+1}^a + \lambda''_a\sigma_j^a\sigma_{j+2}^a] + h\sigma_j^z\right\}. \tag{3.1}$$

This is the most general spin-$\frac{1}{2}$ model with interaction range shorter then two lattice units admitting bond alternations. To ensure translational invariance, we assume that the number of sites N is even. These models can be equivalently represented as nearest-neighbour interactions on a lattice Λ', whose bonds correspond to non-vanishing interactions (see figure 1).

A lattice with such a 'railroad trestle' topology has been considered in [16] (in the case where all the couplings are equal). Notice that the lattice in figure 1 also corresponds

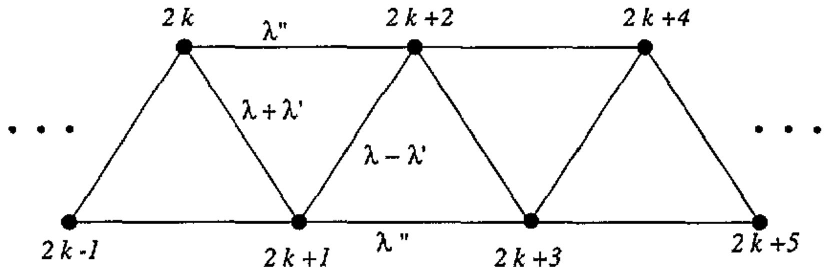

Figure 1. The lattice Λ' corresponding to the Hamiltonian (3.1), whose bonds correspond to non-vanishing interactions.

to a generalization of (3.1) admitting bond alternation for next-to-nearest neighbours. A particular case of such bond alternation yields then the 'sawtooth' topology, which has been shown to possess an exact valence-bond ground state [17].

The models in (3.1) can be equivalently described by a Hamiltonian of the form (2.1). The structure of (2.1) is recovered if we express (3.1) in terms of the variables

$$S_i = \begin{pmatrix} \sigma_{2i-1} \\ \sigma_{2i} \end{pmatrix} \qquad \text{or} \qquad S_i = \begin{pmatrix} \sigma_{2i} \\ \sigma_{2i+1} \end{pmatrix} . \tag{3.2}$$

The family (3.1) contains many interesting systems, including some that are well known integrable models and some which are 'exactly solvable' in some sense. In particular, among *the class of isotropic (globally $su(2)$-invariant) models satisfying $h = 0$ and*

$$\lambda_a = \lambda \qquad \lambda'_a = \lambda' \qquad \lambda''_a = \lambda'' \tag{3.3}$$

for all a, the following special cases are covered by (3.1).

(i) The Heisenberg (XXX) model ($\lambda' = \lambda'' = 0$):

$$H_2 = \sum_{j\in\Lambda} \lambda \sigma_j^a \sigma_{j+1}^a . \tag{3.4}$$

(ii) The staggered XXX model ($\lambda'' = 0$):

$$H_2 = \sum_{j\in\Lambda} [\lambda + \lambda'(-1)^j] \sigma_j^a \sigma_{j+1}^a . \tag{3.5}$$

(iii) The alternating XXX model ($\lambda'' = \lambda = 0$):

$$H_2 = \sum_{j\in\Lambda} \lambda (-1)^j \sigma_j^a \sigma_{j+1}^a . \tag{3.6}$$

(iv) The Majumdar–Ghosh model [18] ($\lambda' = 0, \lambda'' = \frac{1}{2}\lambda$):

$$H_2 = \sum_{j\in\Lambda} [\lambda \sigma_j^a \sigma_{j+1}^a + \tfrac{1}{2} \lambda \sigma_j^a \sigma_{j+2}^a] . \tag{3.7}$$

As is well known this model possesses an exact valence-bond ground state.

For models invariant under global spin rotation around the z-axis,

$$\begin{pmatrix} \sigma_i^x \\ \sigma_i^y \end{pmatrix} \to \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \sigma_i^x \\ \sigma_i^y \end{pmatrix} \tag{3.8}$$

(in which case the global $su(2)$ invariance is broken down to $o(2)$), the coupling constants satisfy

$$\lambda_x = \lambda_y \qquad \lambda'_x = \lambda'_y \qquad \lambda''_x = \lambda''_y . \tag{3.9}$$

Some interesting $o(2)$-symmetric models that can be obtained from a specialization of (3.1) are as follows.

(v) The *XXZ* model ($\lambda'_a = \lambda''_a = 0$):

$$H_2 = \sum_{j\in\Lambda}[\lambda_x(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y) + \lambda_z\sigma_j^z\sigma_{j+1}^z]\,. \tag{3.10}$$

(vi) The Lieb–Schultz–Matis model with alternating Heisenberg and Ising bonds [19]:

$$\begin{aligned} &\lambda''_a = 0 \qquad \lambda_x = \lambda_y = \lambda'_x = \lambda'_y = \tfrac{1}{2} \\ &\lambda_z = (1+U)/2 \qquad \lambda'_z = (1-U)/2 \end{aligned} \tag{3.11}$$

with the Hamiltonian

$$H_2 = \sum_{j\in\Lambda}[\sigma_{2j}^a\sigma_{2j+1}^a + U\sigma_{2j+1}^z\sigma_{2j+2}^z]\,. \tag{3.12}$$

(vii) The Hubbard model:

$$\lambda_x = \lambda_y = \lambda'_x = \lambda'_y = \lambda''_z = 0 \qquad \lambda''_x = \lambda''_y = 1 \qquad \lambda'_z = \pm\,\lambda_z = U \tag{3.13}$$

with the Hamiltonian

$$H_2 = \sum_{j\in\Lambda}[\sigma_j^+\sigma_{j+2}^- + \sigma_j^-\sigma_{j+2}^+ + U[1+(-1)^j]\sigma_j^z\sigma_{j+1}^z]\,. \tag{3.14}$$

The equivalence of (3.14) (for a lattice with an even number of sites) with the usual formulation of the Hubbard model

$$H = \sum_{j\in\Lambda}[s_j^x s_{j+1}^x + s_j^y s_{j+1}^y + t_j^x t_{j+1}^x + t_j^y t_{j+1}^y + U s_j^z t_j^z] \tag{3.15}$$

(where s_i and t_i are two independent sets of Pauli matrices at site i) can be seen by redefining the spin variables in (3.14) as

$$s_j^a \to \sigma_{2j}^a \qquad t_j^a \to \sigma_{2j+1}^a\,. \tag{3.16}$$

3.2. *Strategy of the test*

For the completely general anisotropic case, a natural candidate for the third-order charge involves only nearest-neighbour interactions on the the lattice in figure 1 and has the form

$$\begin{aligned} H_3 = \sum_{j\in\Lambda}\sum_{a_i=x,y,z} \big[&(\alpha^{(1)}_{a_1a_2a_3} + (-1)^j\alpha'^{(1)}_{a_1a_2a_3})\sigma_j^{a_1}\sigma_{j+1}^{a_2}\sigma_{j+2}^{a_3} \\ &+(\alpha^{(2)}_{a_1a_2a_3} + (-1)^j\alpha'^{(2)}_{a_1a_2a_3})\sigma_j^{a_1}\sigma_{j+2}^{a_2}\sigma_{j+3}^{a_3} \\ &+(\alpha^{(3)}{}_{a_1a_2a_3} + (-1)^j\alpha'^{(3)}_{a_1a_2a_3})\sigma_j^{a_1}\sigma_{j+1}^{a_2}\sigma_{j+3}^{a_3} \\ &+(\alpha^{(4)}_{a_1a_2a_3} + (-1)^j\alpha'^{(4)}_{a_1a_2a_3})\sigma_j^{a_1}\sigma_{j+2}^{a_2}\sigma_{j+4}^{a_3}\big] \end{aligned} \tag{3.17}$$

where $\alpha^{(i)}_{a_1a_2a_3}$ and $\alpha'^{(i)}_{a_1a_2a_3}$ are arbitrary coefficients. We search for integrable systems by imposing the condition of commutativity of H_3 with the Hamiltonian. This leads to an over-determined system of equations for the set of parameters in H_3. This system has non-trivial solutions only for special values of the parameters of the Hamiltonian. However, the analysis of this system is very cumbersome, since in the absence of additional symmetries, (3.17) contains 216 free parameters! Henceforth, we consider only three special situations, in which the analysis simplifies significantly: the isotropic case, the $o(2)$-symmetric case, and the anisotropic case without bond alternation nor next-to-nearest neighbour interaction, i.e. the *XYZh* model.

One might consider other candidate charges, involving triples of sites other than those in (3.17). However, suppose that one cluster different from those appearing in (3.17) is introduced. Then (considering an infinite chain), to cancel the new terms arising in the commutator with the Hamiltonian, an infinite sequence of other clusters would have to be added, with the distance between spins in such clusters growing arbitrarily. This would violate the requirement of locality. (The above reasoning is not so obvious, however, when some of the couplings vanish.)

3.3. The XYZh model

We first consider (3.1) in the absence of bond alternation and next-to-nearest neighbour terms. The most general candidate charge H_3 coupling three nearest-neighbour spins is

$$H_3 = \sum_{j\in\Lambda} \sum_{a_i=x,y,z} \alpha_{a_1a_2a_3} \sigma_j^{a_1} \sigma_{j+1}^{a_2} \sigma_{j+2}^{a_3} \tag{3.18}$$

which contains 27 free parameters. By enforcing the commutativity of this candidate H_3 with the Hamiltonian, we find that if $h \neq 0$ and no two couplings are equal, the only solution is $\alpha_{a_1a_2a_3} = 0$ for all triples $a_1a_2a_3$. Thus there is no non-trivial charge H_3 for the anisotropic *XYZ* chain in a non-zero magnetic field. This suggests thus that the *XYZh* model is non-integrable, in agreement with the fact that the Bethe ansatz solution for the *XYZ* model cannot be generalized to the case with a non-trivial magnetic field.

3.4. The isotropic case

In the isotropic case, the most general third-order charge involving only the nearest neighbours on the lattice in figure 1 has the following form:

$$\begin{aligned} H_3 = \sum_{j\in\Lambda} \epsilon_{abc} [(\alpha + (-1)^j \alpha')\sigma_j^a \sigma_{j+1}^b \sigma_{j+2}^c + (\beta + (-1)^j \beta')\sigma_j^a \sigma_{j+2}^b \sigma_{j+3}^c \\ + (\tilde{\beta} + (-1)^j \tilde{\beta}')\sigma_j^a \sigma_{j+1}^b \sigma_{j+3}^c + (\gamma + (-1)^j \gamma')\sigma_j^a \sigma_{j+2}^b \sigma_{j+4}^c] \end{aligned} \tag{3.19}$$

where $\alpha, \beta, \tilde{\beta}, \gamma$ and their primed variants are arbitrary parameters. Solving the commutativity condition $[H_2, H_3] = 0$, we find that (for $N \geqslant 8$) non-trivial solutions (for which not all parameters of H_3 are equal to zero) exist only in two cases: for the *XXX* model ($\lambda' = \lambda''$), with the solution $\alpha \neq 0$, $\beta = \beta' = \tilde{\beta} = \tilde{\beta}' = \gamma = \gamma' = 0$, i.e.

$$H_3 = \sum_{j\in\Lambda} \alpha \epsilon_{abc} \sigma_j^a \sigma_{j+1}^b \sigma_{j+2}^c \tag{3.20}$$

and for $\lambda = \lambda'$, which corresponds to two decoupled *XXX* models on the even and odd sublattices (the solution is then $\gamma, \gamma' \neq 0$ and all other parameters of H_3 equal to zero).

These results suggest that the only integrable isotropic models within the family (3.1) are of the *XXX* type. In particular, the Majumdar–Ghosh model, the alternating *XXX* model, and the staggered *XXX* model† all fail the test of the existence of H_3 and thus seem to be non-integrable.

† Note that in the continuous limit the alternating term corresponds to tr g in the WZNW model [20] with a level-one affine $su(2)$ spectrum generating algebra. g stands for the basic field in the WZNW model, a 2×2 matrix in the $su(2)$ case. tr g turns out to be an integrable perturbation of this WZNW model [21]. However, the staggered *XXX* model in the continuous limit gives the WZNW model perturbed by both tr g and the marginal current–current term. Together, these two perturbations are incompatible with integrability. Note also that for $\lambda = \pm\lambda'$ the staggered *XXX* model degenerates into a pathological 'chopped *XXX*' chain consisting of disjoint bonds (see section 3.5 for a discussion of a similar case).

For $N < 8$ there exist non-trivial solutions for H_3 of the form (3.19). Let us illustrate this in the case $\lambda'_x = 0$, when the Hamiltonian (3.1) contains only nearest and next-to-nearest neighbours:

$$H_2 = H + \mu H' \tag{3.21}$$

where μ is an arbitrary parameter, and H and H' are given by (2.5) and (2.7), respectively. For $N = 5$ we find that the *XXX* charge (3.20) commutes with (3.21). This is a consequence of the 'accidental' identity (2.9) holding for $N = 5$. In other words, for $N = 5$ the next-to-nearest interaction is a 'non-local' charge (related to the square of the total spin), which commutes with all the *XXX* charges. For $N > 5$ this solution disappears (the three-spin *XXX* charge no longer commutes with the Hamiltonian (3.21)). Similarly, for $N = 6$ there is a one-parameter family of solutions

$$H_3 = H_3^{XXX} + \nu F_{3,1} - \tfrac{1}{3}(\mu + \mu\nu - \nu) F^s_{3,2} \tag{3.22}$$

where ν is an arbitrary parameter, and†

$$F_{3,1} = \sum_{j\in\Lambda} \epsilon^{abc}[\sigma_j^a \sigma_{j+2}^b \sigma_{j+3}^c + \sigma_j^a \sigma_{j+1}^b \sigma_{j+3}^c] \tag{3.23}$$

$$F^s_{3,2} = \sum_{j\in\Lambda} \epsilon^{abc} \sigma_j^a \sigma_{j+2}^b \sigma_{j+4}^c . \tag{3.24}$$

For $N = 7$ there is another solution:

$$H_3 = H_3^{XXX} + \mu F_{3,1} - \mu^2 F^s_{3,2} . \tag{3.25}$$

Similar accidental non-local three-spin charges, whose form changes with N, exist in fact for all N, but it is only for $N < 8$ that they 'look local', i.e. can be put in the form (3.19).

3.5. *The $o(2)$-invariant case*

We are searching again for a non-vanishing charge H_3 of the form (3.17). The requirement of the $o(2)$ invariance imposes a number of restrictions on the parameters $\alpha^{(i)}_{a_1a_2a_3}$ and $\alpha'^{(i)}_{a_1a_2a_3}$:

$$\begin{aligned}\alpha_{yxy} = \alpha_{zxz} = \alpha_{xyx} = \alpha_{zyz} = \alpha_{yyx} = \alpha_{xyy} = \alpha_{zzx} = \alpha_{xzz} = \alpha_{zzy} = \alpha_{yzz} = \alpha_{xxy} \\ = \alpha_{yxx} = \alpha_{xxx} = \alpha_{yyy} = 0\end{aligned} \tag{3.26}$$

and

$$\alpha_{yxz} = -\alpha_{xyz}, \alpha_{zyx} = -\alpha_{zxy}, \alpha_{xzy} = -\alpha_{yzx}\alpha_{yzy} = \alpha_{xzx}, \alpha_{zxx} = \alpha_{zyy}, \alpha_{xxz} = \alpha_{yyz} . \tag{3.27}$$

As a result, the number of free parameters in H_3 is now decreased to 56. Again one is looking for the values of these parameters allowing for the existence of a non-trivial solution (persisting when N is increased). This system is best analysed with a computer. The results are given below‡.

If λ''_x is not zero, a non-trivial solution for H_3 exists only in two cases:

(i) for two decoupled *XXZ* models on two disjoint (even and odd) sublattices: ($\lambda_x = \lambda'_x = \lambda_z = \lambda'_z$); and

† $F^s_{3,2}$ is the symmetric part of the quantity $F_{3,2}$ introduced in [15, 22].

‡ Note that trivial $o(2)$-symmetric models with Hamiltonians involving only the z-components of the spin variables at each site are explicitly excluded from consideration.

(ii) the Hubbard model: ($\lambda_x = \lambda'_x = \lambda''_z = 0, \lambda_z = \pm\lambda'_z$). The solution for the charge H_3 is then

$$H_3 = \sum_{j\in\Lambda}[2\lambda'_z(\sigma_j^y\sigma_{j+1}^z\sigma_{j+2}^x - \sigma_j^x\sigma_{j+1}^z\sigma_{j+2}^y) + \lambda''_x(\sigma_j^y\sigma_{j+2}^z\sigma_{j+4}^x - \sigma_j^x\sigma_{j+2}^z\sigma_{j+4}^y) \\ +\lambda'_z(1-(-)^j)(\sigma_j^x\sigma_{j+2}^y\sigma_{j+3}^z - \sigma_j^y\sigma_{j+2}^x\sigma_{j+3}^z \\ -\sigma_j^z\sigma_{j+1}^y\sigma_{j+3}^x + \sigma_j^z\sigma_{j+1}^x\sigma_{j+3}^y)] \tag{3.28}$$

which can be translated, using (3.16), into the usual expression for the third-order charge in the Hubbard model [8, 15].

If λ''_x is zero there are more possibilities. non-trivial solutions for H_3 exist in four cases. These are:

(iii) the *XXZ* model: $\lambda'_x = \lambda'_z = \lambda''_z = 0$;

(iv) the staggered *XX* model: $\lambda_z = \lambda'_z = \lambda''_z = 0$;

(v) the staggered *XXZ* model† $\lambda_x = \lambda'_z = \lambda''_z = 0$; and

(vi) the model with alternating *XXZ* and Ising bonds: $\lambda_x = \pm\lambda'_x$.

The three-spin charge obtained for the *XXZ* model is

$$H_3 = \sum_{j\in\Lambda}[\lambda_z(\sigma_j^x\sigma_{j+1}^y\sigma_{j+2}^z - \sigma_j^y\sigma_{j+1}^x\sigma_{j+2}^z + \sigma_j^z\sigma_{j+1}^x\sigma_{j+2}^y - \sigma_j^z\sigma_{j+1}^y\sigma_{j+2}^x) \\ +\lambda_x(\sigma_j^x\sigma_{j+1}^z\sigma_{j+2}^y - \sigma_j^y\sigma_{j+1}^z\sigma_{j+2}^x)] \tag{3.29}$$

in agreement with [6, 15].

For the staggered *XX* model [24]

$$H_2 = \sum_{j\in\Lambda}(\lambda + \lambda'(-1)^j)(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y) \tag{3.30}$$

the three-spin charge obtained via our test is identical to the *XX* charge, that is

$$H_3 = \sum_{j\in\Lambda}\alpha(\sigma_j^x\sigma_{j+1}^z\sigma_{j+2}^x + \sigma_j^y\sigma_{j+1}^z\sigma_{j+2}^y) + \beta(\sigma_j^x\sigma_{j+1}^z\sigma_{j+2}^y - \sigma_j^y\sigma_{j+1}^z\sigma_{j+2}^x) \tag{3.31}$$

where α and β are arbitrary coefficients. Another indication of integrability of this model is provided by its continuum limit, which corresponds to a theory of free massive fermions [25]. We present in appendix B a simple direct proof of the integrability of the lattice model (3.30) by exhibiting a family of mutually commuting conservation laws.

For the staggered *XXZ* model, defined by the Hamiltonian

$$H_2 = \sum_{j\in\Lambda}[\lambda'_x(-1)^j(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y) + \lambda_z\sigma_j^z\sigma_{j+1}^z] \tag{3.32}$$

the three-spin charge obtained from the test has the following form:

$$H_3 = \sum_{j\in\Lambda}[\lambda_z(-1)^j(\sigma_j^x\sigma_{j+1}^y\sigma_{j+2}^z - \sigma_j^y\sigma_{j+1}^x\sigma_{j+2}^z + \sigma_j^z\sigma_{j+1}^x\sigma_{j+2}^y - \sigma_j^z\sigma_{j+1}^y\sigma_{j+2}^x) \\ +\lambda_x'(\sigma_j^x\sigma_{j+1}^z\sigma_{j+2}^y - \sigma_j^y\sigma_{j+1}^z\sigma_{j+2}^x)]\,. \tag{3.33}$$

This charge can be also obtained using the boost operator as $[B, H_2]$. Note that the transformation

$$\sigma_{2j+1}^x \to \sigma_{2j+1}^y \qquad \sigma_{2j+1}^y \to -\sigma_{2j+1}^x \qquad \sigma_{2j}^x \to \sigma_{2j}^x \qquad \sigma_{2j}^y \to \sigma_{2j}^y \tag{3.34}$$

† This system is related to the two-dimensional Ashin–Teller model, see [23].

which corresponds to a spin rotation by $\pi/2$ around the z-axis restricted to odd sites, establishes the equivalence of the staggered *XXZ* chain with the model

$$H_2 = \sum_{j\in\Lambda}[\sigma_j^x\sigma_{j+1}^y - \sigma_j^y\sigma_{j+1}^x + \lambda_z\sigma_j^z\sigma_{j+1}^z]. \tag{3.35}$$

This is a particular case of the *XXZ* model with Dzyaloshinski–Moriya interaction [26]:

$$H_2 = \sum_{j\in\Lambda} J_x(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y) + D(\sigma_j^x\sigma_{j+1}^y - \sigma_j^y\sigma_{j+1}^x) + J_z\sigma_j^z\sigma_{j+1}^z \tag{3.36}$$

where J_x, J_z and D are arbitrary parameters. Integrability of (3.32) follows then from the integrability of (3.36), which has been proven in [27]. Note also that the model (3.36) is not $o(2)$-invariant, and, by a spin rotation (3.8) with a suitably chosen angle α, it may be transformed into the anisotropic Dzyaloshinski–Moriya system:

$$H_2 = \sum_{j\in\Lambda}[D_x\sigma_j^x\sigma_{j+1}^y + D_y\sigma_j^y\sigma_{j+1}^x + D_z\sigma_j^z\sigma_{j+1}^z]. \tag{3.37}$$

The model with alternating *XXZ* and Ising bonds

$$\begin{aligned}H_2 = \sum_{j\in\Lambda}[(\lambda_x + \lambda_x')(\sigma_{2j}^x\sigma_{2j+1}^x + \sigma_{2j}^y\sigma_{2j+1}^y) + (\lambda_z + \lambda_z')\sigma_{2j}^z\sigma_{2j+1}^z \\ +(\lambda_z - \lambda_z')\sigma_{2j+1}^z\sigma_{2j+2}^z + \lambda_z''\sigma_j^z\sigma_{j+2}^z]\end{aligned} \tag{3.38}$$

which is a slight generalization of the Lieb–Schultz–Mattis model, presents certain peculiarities. The model has been diagonalized in [19] (for $\lambda''_z = 0, \lambda_x = \lambda'_x = \frac{1}{2}, \lambda_z = (1+U)/2, \lambda'_z = (1-U)/2$). As observed in [19], a convenient basis is provided by the eigenstates of $\boldsymbol{L}_j = \boldsymbol{\sigma}_{2j} + \boldsymbol{\sigma}_{2j+1}$. Consider then subspaces of the space of states corresponding to a particular sequence $\{M_j\}$ of the eigenvalues of the third component of the $\boldsymbol{L}_j$. Since the Lieb–Schultz–Mattis-type Hamiltonian (3.38) commutes with each of the $\boldsymbol{L}_j$ it does not mix different subspaces; in other words it is block-diagonal in this basis, and can be diagonalized separately in each subspace. The projection operators onto the subspaces corresponding to different sequences $\{M_j\}$ provide then a mutually commuting set of operators, all commuting with the Hamiltonian. Therefore, (3.38) satisfies the definition of integrability given in the introduction; admittedly the nature of these conserved charges appears a bit unusual and somewhat trivial.

The explicit form H_3 charge found via the the test is

$$H_3 = \sum_{j\in\Lambda}[\alpha L_j^z\sigma_{2j+2}^z\sigma_{2j+3}^z + \beta\sigma_{2j}^z\sigma_{2j+1}^z L_{j+1}^z] \tag{3.39}$$

where α and β are arbitrary coefficients. Interestingly, not only does the above sum commute with (3.38), but each term in it is separately conserved. Clearly, all these terms can be expressed as linear combinations of the projection operators discussed above. Let $P_j^{(m_j)}$ denote a projection onto the states with $M_j = m_j$. Then $\sigma_{2j}^z\sigma_{2j+1}^z = \frac{1}{4}(P_j^{(1)} + P_j^{(-1)} - P_j^{(0)})$ and $L_j^z = P_j^{(1)} - P_j^{(-1)}$. In particular, in the $2^{N/2}$-dimensional subspace where all the M_i are zero (which is the sector containing the vacuum), (3.39) vanishes.

The block-diagonal nature of the Hamiltonian (and hence the existence of the set of commuting projections) is manifestly preserved by the addition to (3.38) of an arbitrary interaction involving only the z-components of the spin variables. Finally, we note that a particular case of (3.38) with $\lambda_z - \lambda_z' = \lambda_z'' = 0$ provides a pathological 'chopped *XXZ*' system, consisting of $N/2$ disjoint *XXZ* bonds, with all neighbouring links trivially commuting.

Summing up, it appears that apart from the models (i)–(vi) described above, all other $o(2)$-symmetric models within the family (3.1) are non-integrable. Among the integrable models, there are three situations in which the first moment of the Hamiltonian (the boost) acts as a ladder operator for conservation laws: the XXZ chain (cases (i) and (iii)), the staggered XX model (iv), and the staggered XXZ chain (v).

4. Example 2: isotropic spin-1 chains

Consider now a class of isotropic spin-1 chains with nearest-neighbour interactions. The most general Hamiltonian contains a bilinear and a biquadratic term

$$H_2(\beta) = \sum_{j\in\Lambda} [S_j^a S_{j+1}^a + \beta (S_j^a S_{j+1}^a)^2] \tag{4.1}$$

where the S_j^a are the $su(2)$ spin-1 matrices, acting non-trivially only on the jth factor of the Hilbert space $\bigotimes_j \mathbf{C}^3$. For convenience we choose the representation in which S^z is diagonal, i.e.

$$S^x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad S^y = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad S^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{4.2}$$

Using the identity

$$(S_j^a S_k^a)^2 = \tfrac{1}{4} D_j^{ab} D_k^{ab} - \tfrac{1}{2} S_j^a S_k^a \tag{4.3}$$

where

$$D_j^{ab} \equiv S_j^a S_j^b + S_j^b S_j^a \tag{4.4}$$

the Hamiltonian can expressed as

$$H_2(\beta) = \sum_{j\in\Lambda} \big[(1 - \beta\tfrac{1}{2}) S_j^a S_{j+1}^a + \beta \tfrac{1}{4} D_j^{ab} D_{j+1}^{ab} \big]. \tag{4.5}$$

The boost operator yields the following candidate for the H_3 charge:

$$\begin{aligned} H_3(\beta) = \sum_{j\in\Lambda} \epsilon^{abc} \big[&(-2 + 2\beta - \beta^2\tfrac{1}{2}) S_j^a S_{j+1}^b S_{j+2}^c - \beta^2 \tfrac{1}{2} D_j^{ad} S_{j+1}^b D_{j+2}^{cd} \\ &+ (-\beta + \beta^2\tfrac{1}{2})(D_j^{ad} D_{j+1}^{db} S_{j+2}^c + S_j^a D_{j+1}^{bd} D_{j+2}^{dc}) \big]. \end{aligned} \tag{4.6}$$

The commutator $[H_2(\beta), H_3(\beta)]$ vanishes only for $\beta = \pm 1$ or $\beta = \infty$; these cases have already been identified as integrable in the literature. For $\beta = -1$, (4.1) reduces to the isotropic version of the Fateev–Zamolodchikov chain, associated with the 19-vertex model [28], whose integrability follows directly from the Yang–Baxter equation. For $\beta = 1$, equation (4.1) describes the Sutherland $su(3)$ symmetric chain [29], whose Hamiltonian can be rewritten in terms of the Gell-Mann matrices t^a:

$$H_2(1) \sim \sum_{j\in\Lambda} t_j^a t_{j+1}^a. \tag{4.7}$$

It can be solved by the nested Bethe ansatz [29] and is also related directly to the Yang–Baxter equation. Note that in this case, $H_3(1)$ can also be written in the form [15]

$$H_3 = \sum_{j\in\Lambda} f^{abc} t_j^a t_{j+1}^b t_{j+2}^c. \tag{4.8}$$

In the limit $\beta \to \infty$, (4.1) reduces to a system with purely biquadratic interactions, whose integrability has previously been established in [30].

For the spin-1 models, (4.1) with finite $\beta \neq \pm 1$, the boost operator does not produce a conserved quantity. Furthermore, as we show below, there exists no non-trivial local conserved charge involving up to three nearest neighbours. The most general expression for such a charge would have the form

$$H_3 = \sum_{j\in\Lambda}[a_1\epsilon^{abc}S_j^a S_{j+1}^b S_{j+2}^c + a_2\epsilon^{abc}D_j^{ad}D_{j+1}^{db}S_{j+2}^c + a_3\epsilon^{abc}S_j^a D_{j+1}^{bd}D_{j+2}^{dc}$$
$$+a_4\epsilon^{abc}D_j^{ad}S_{j+1}^b D_{j+2}^{cd} + a_5 D_j^{ab}D_{j+1}^{bc}D_{j+2}^{ac} + a_6 S_j^a S_{j+1}^b D_{j+2}^{ab}$$
$$+a_7 D_j^{ab}S_{j+1}^a S_{j+2}^b + a_8 S_j^a D_{j+1}^{ab} S_{j+2}^b + a_9 S_j^a S_{j+1}^a + a_{10}(S_j^a S_{j+1}^a)^2]. \tag{4.9}$$

where the a_i are undetermined coefficients. Enforcing the commutativity of H_3 with the Hamiltonian, we obtain as usual a number of constraints on these coefficients. In particular, the vanishing of the four-spin terms in this commutator requires

$$a_2 - a_3 = a_5 = a_6 = a_7 = a_8 = 0 \tag{4.10}$$

and

$$\begin{aligned} &a_2 = a_4 = 0 \qquad \text{if} \quad \beta = 0 \\ &a_1 = (1 + 4/\beta^2 - 4/\beta)a_4 \qquad a_2 = (-1 + 2/\beta)a_4 \qquad \text{if} \quad \beta \neq 0. \end{aligned} \tag{4.11}$$

The above conditions means that in order for H_3 to commute with H_2, its three-spin part must be proportional to the commutator of the boost and H_2 (both for $\beta = 0$ and $\beta \neq 0$). The vanishing of the terms with two and three spins in the commutator imposes two further restrictions. First, the two-spin part of H_3 must be proportional to the Hamiltonian H_2. Second, unless $\beta^2 = 1$ or $\beta = \infty$, the three-spin part of H_3 must be trivial.

The non-existence of H_3 for (4.1) with finite $\beta \neq \pm 1$, suggests that all these models are non-integrable. In particular, the bilinear system ($\beta = 0$), as well as the one with $\beta = 1/3$, for which there exists an exact valence-bond ground state [31], all fail the above test.

It should be added that the necessary condition (2.21) for having a quantum chain related to a solution of the Yang–Baxter equation has been examined for isotropic spin-s chains with $s < 14$ in [14, 32]. In particular, these authors found that the only spin-1 systems satisfying (2.21) are $\beta = \pm 1$ and $\beta = \infty$. As we have discussed above, this implies that for other values of β the Hamiltonian (4.1) cannot be a fundamental model. But *a priori* there might exist integrable but non-fundamental models for $\beta \neq \pm 1, \infty$. Our results, showing that there is no non-trivial three-spin charge, provide much stronger evidence for the non-integrability of all the isotropic spin-1 models with $\beta \neq \pm 1, \infty$. It would be interesting to perform a similar analysis for $s > 1$; such analysis is however much more complicated for higher s.

We end this section with a short remark on the general spin-s bilinear system. Is it possible that this system is, by a bizarre accident, an integrable fundamental model for some values of s? This has been answered negatively in [32] for all $s < 100$, using computer algebra. Here we present a simple calculation showing that for the bilinear systems, the boost operator can never produce a conserved quantity for $s \neq \frac{1}{2}$, which thus excludes the possibility of such an accident. Acting on the Hamiltonian, the boost operator generates a candidate for the three-spin charge of the form

$$H_3 = \sum_{j\in\Lambda}\epsilon^{abc}S_j^a S_{j+1}^b S_{j+2}^c. \tag{4.12}$$

The commutator of this quantity with the Hamiltonian is then

$$[H_3, H_2] = \sum_{j\in\Lambda}[D_j^{ab}S_{j+1}^a S_{j+2}^b - D_j^{aa}S_{j+1}^b S_{j+2}^b + S_j^a S_{j+1}^a D_{j+2}^{bb} - S_j^a S_{j+1}^b D_{j+2}^{ab}]. \tag{4.13}$$

The vanishing of this sum requires that all the terms containing spins in an arbitrary cluster vanish separately, which is not the case in general. The sum can vanish only if the terms cancel two by two, which is possible only if $D^{ab} \sim \delta^{ab}$, a condition which is not true unless

$s = \frac{1}{2}$. Therefore, the bilinear Heisenberg chain is an integrable fundamental model only for $s = \frac{1}{2}$.

5. Example 3: Potts models

In this section we consider the class of Hamiltonians

$$H_2 = \sum_{j\in\Lambda}\sum_{k=1}^{N-1}[\alpha_k Z_j^k Z_{j+1}^{N-k} + \beta_k X_j^k] \tag{5.1}$$

where Z_j and X_j are $N\times N$ matrices at site j with entries $(Z_j)_{mn} = \exp[2\mathrm{i}\pi(n-1)/N]\delta_{m,n}$, $(X_j)_{mn} = \delta_{m,n+1}$. This class of Hamiltonians contains, for a particular choice of the parameters, the integrable chiral Potts model [33]:

$$\alpha_k = \frac{\exp[\mathrm{i}\phi(2k/N-1)]}{\sin(\pi k/N)} \qquad \beta_k = \lambda\frac{\exp[\mathrm{i}\psi(2k/N-1)]}{\sin(\pi k/N)} \tag{5.2}$$

with

$$\cos\phi = \lambda\cos\psi\,. \tag{5.3}$$

Note that there is also another integrable model within the class (5.1), corresponding to the choice $\alpha_2 = \beta_2 = 0$ (which can be in fact generalized to the inhomogeneous case [34]). For $\phi = \psi = \pi/2$, (5.2) reduces to the superintegrable model of Gehlen and Rittenberg [35]. In this case, there exists a recursive construction of conserved charges based on the Onsager algebra [36] generalizing the Dolan–Grady [10] condition. In particular, the three-point charge in the superintegrable case is given by

$$H_3 = [H,[\tilde{H},H]] + [\tilde{H},[H,\tilde{H}]] + N^2(\lambda H + \tilde{H}) \tag{5.4}$$

where, in the notation of appendix A,

$$H_2 = H + \lambda\tilde{H} \tag{5.5}$$

with $H = \sum_{j\in\Lambda}\sum_{k=1}^{N-1}\alpha_k Z_j^k Z_{j+1}^{N-k}$, and $\tilde{H} = \sum_{j\in\Lambda}\sum_{k=1}^{N-1}\alpha_k X_j^k$.

In the following we will investigate (5.1) in the case where none of the coupling constants vanish: $\alpha_k, \beta_k \neq 0$. With H_3 as a candidate, we will consider a multiparameter modification of (5.4), containing the same types of terms, but now with arbitrary coefficients. For $N = 3$ this leads to

$$\begin{aligned} H_3 = \sum_{j\in\Lambda}\sum_{k=1}^{N-1}[& a_k^{(1)}(Z_jY_{j+1}Z_{j+2})^k + a_k^{(2)}(Z_jY_{j+1}^*Z_{j+2})^k + a_k^{(3)}Z_j^kX_{j+1}Z_{j+2}^{N-k} \\ & + a_k^{(4)}Z_j^kX_{j+1}^2Z_{j+2}^{N-k} + a_k^{(5)}Z_j^kZ_{j+1}^{N-k} + a_k^{(6)}Z_j^kY_{j+1}^{N-k} + a_k^{(7)}Y_j^kZ_{j+1}^{N-k} \\ & + a_k^{(8)}(Z_jY_{j+1}^*)^k + a_k^{(9)}(Y_j^*Z_{j+1})^k + a_k^{(10)}Y_j^kY_{j+1}^{N-k} + a_k^{(11)}Y_j^{*k}Y_{j+1}^{*N-k} \\ & + a_k^{(12)}(Y_jY_{j+1}^*)^k + a_k^{(13)}(Y_j^*Y_{j+1})^k + a_k^{(14)}X_j^k] \end{aligned} \tag{5.6}$$

where $a_k^{(i)}$ are arbitrary coefficients and the $N\times N$ matrix Y_j is defined as $Y_j = X_jZ_j$. Calculating the commutator of this candidate H_3 with the Hamiltonian (5.1), we find that a non-trivial solution for H_3 exists only if the parameters of the model satisfy the relation

$$\frac{(\alpha_1^3+\alpha_2^3)}{\alpha_1\alpha_2} = \frac{(\beta_1^3+\beta_2^3)}{\beta_1\beta_2}\,. \tag{5.7}$$

As shown in [33], the most general solution of this relation (modulo symmetry transformations in the space of couplings) is given by (5.2). Therefore, the $N = 3$ chiral

Potts model satisfies conjecture 1. The same conclusion may also be obtained for $N > 3$. The non-existence of non-trivial solutions to $[H_2, H_3] = 0$ when (5.7) is not satisfied, suggests that the chiral Potts model is the only integrable system within the $N = 3$ family (5.1) with couplings $\alpha_1, \alpha_2, \beta_1, \beta_2 \neq 0$. However, to rule out a possibility of other integrable points within (5.1) would require an ansatz for a three-point charge more general then (5.6); such an expression, even for $N = 3$, may contain several hundred free parameters, making the analysis extremely cumbersome.

Note that a ladder operator for the Potts model (5.1) exists only in the critical case $\alpha_k = \beta_k = 1$, where the infinite chain Hamiltonian can be written down, modulo constants, as

$$H_2 = \sum_{\ell \in \mathbb{Z}+1/2} U_\ell \tag{5.8}$$

where

$$U_j = \sum_{k=0}^{N-1} (\omega^m X_j)^k \qquad U_{j+1/2} = \sum_{k=0}^{N-1} (\omega^m Z_j Z_{j+1}^{N-1})^k \qquad j \in \mathbf{Z} \tag{5.9}$$

with $\omega = \exp[2i\pi/N]$ and m an arbitrary integer. (The operators U_ℓ generate a Temperley–Lieb algebra.) In this case the three-spin charge H_3 can be obtained by the action of the first moment of the Hamiltonian density as $[B, [B, H_2]]$†.

6. Example 4: Inhomogeneous spin-$\frac{1}{2}$ models

Conjectures 1 and 2 have been formulated for a class of models with translationally invariant (i.e. periodic) nearest-neighbour interactions. As already mentioned, many models which originally do not satisfy this requirement may be reduced to such a form by grouping together spins at several neighbouring sites in a single vector-like spin variable. However, for completely inhomogeneous models, periodicity of the interaction cannot be recovered by this procedure. Nevertheless, the H_3 test seems to remain valid for such systems. In this section, this will be illustrated in the context of the totally inhomogeneous Ising model [37].

We consider the following class of Hamiltonians:

$$H_2 = \sum_{j \in \Lambda} [u_j \sigma_j^x \sigma_{j+1}^x + h_j \sigma^z] \tag{6.1}$$

where u_j and h_j are arbitrary constants; we will assume that all of them are non-vanishing. We look for a three-spin charge of the form

$$H_3 = \sum_{j \in \Lambda} [a_j \sigma_j^x \sigma_{j+1}^z \sigma_{j+2}^x + b_j \sigma_j^y \sigma_{j+1}^y + c_j \sigma_j^x \sigma_{j+1}^x + d_j \sigma_j^z] \,.$$

(When the couplings are j-independent this charge reduces to the three-spin charge of the homogeneous Ising model.) The vanishing of the commutator $[H_2, H_3]$ leads then to the following relations:

$$\begin{aligned}
& a_{j-1} h_{j+1} + b_j u_{j-1} = 0 \\
& a_j h_j + b_j u_{j+1} = 0 \\
& -a_j u_{j+1} - b_j h_j + c_j h_{j+1} - d_{j+1} u_j = 0 \\
& -a_{j-1} u_{j-1} - b_j h_{j+1} + c_j h_j - d_j u_j = 0 \,.
\end{aligned} \tag{6.2}$$

† Note that $[B, H_2]$ is a two-spin charge (which is obviously different from the Hamiltonian (5.8)), commuting with all the other charges. A deformed version of this charge is also conserved for the general chiral Potts model.

The first two equations are solved by taking

$$a_{j+1} = a_j \frac{h_{j+2}u_{j+2}}{h_{j+1}u_j} \tag{6.3}$$

$$b_j = -a_{j-1}h_{j+1}/u_{j-1}\,. \tag{6.4}$$

Equation (6.3) allows for the successive determination of the a_j for all points of the lattice starting from some given site, e.g. from a_1. For a finite chain, one must also check the compatibility of this solution with periodic boundary conditions. For a chain of M sites, the first $M-1$ equations in (6.3) suffice to express $\{a_2, a_3, \ldots, a_M\}$ as a function of a_1. In particular, one obtains $a_M = a_1 h_{M+1} u_M/(h_2 u_2)$, which is indeed compatible with the last equation ($j = M$) in (6.3) since $h_{M+1} = h_1$.

Substitution of (6.3) and (6.4) in (6.2) yields a system of $2M$ linear equations with $2M$ unknowns $\{c_j, d_j\}$. Since the determinant of this system is $(-)^M \prod_{j=1}^{M} u_j h_j$, the solution always exists, provided that none of the couplings vanishes. Therefore, there is a non-trivial three-spin charge for the totally inhomogeneous model (6.1). This is not surprising as this model is well known to be integrable, being equivalent to a free fermion system [24].

One could also investigate a more general class of Hamiltonians, i.e. totally inhomogeneous *XYZh* models. In the truly inhomogeneous case, a non-trivial solution for H_3 may be found only for the inhomogeneous Ising model (6.1) and the closely related inhomogeneous *XY* model. This suggests that these are the only integrable models in this class.

7. Concluding remarks

The simple integrability test considered in this work appears to be applicable to a rather general class of quantum chains. However, the range of applicability of the conjectures 1 and 2 has not yet been determined rigorously. Regardless of that, this simple method seems to be a useful heuristic tool.

Here the test has been applied to several types of models. Even though no new integrable points have been found, the positive aspects of this analysis should be emphasized. First of all, the test has been applied to a class of models that has been extensively studied in the past and it would have been surprising to find many new integrable systems. Our simple criterion allowed us to recover rather easily all the known integrable cases. Futhermore, apart from those integrable chains identified here, it strongly indicates the non-existence of other integrable models among those falling within the class studied.

It may be worthwhile to use the test to try to identify integrable models within other physically relevant families of models. In particular, the completely anisotropic case of the 10-parameter family considered in section 3 remains to be studied in detail. However, we do not expect that it will reveal new types of integrable systems, beyond anisotropic generalizations of the systems found in the $o(2)$-symmetric case. In particular, we expect that the *XYZ* chain, the staggered *XY* model, the staggered *XYZ* chain (with the xy part alternating in sign) equivalent to the Dzyaloshinski–Moriya system (3.37), a generalized Lieb–Schultz–Mattis model, and a generalized Hubbard model (consisting of two copies of the *XY* chain interacting along their z components) are, up to a relabelling of variables, the only non-trivial integrable anisotropic models within the family (3.1). Another interesting area of application is provided by the $SU(N)$-invariant chains. Recently, all the isotropic $SU(3)$-invariant chains satisfying the Reshetikhin condition have been identified in [38]. It would be interesting to determine if these are the only ones with non-vanishing H_3.

The present analysis focuses on models whose spin variables define an associative algebra. If one admits non-associative algebras, there exist infinite chains with an infinite number of conservation with higher-order charges not commuting among themselves. The octonionic chain found recently [39] is an example of such a chain. But because the mutual non-commutativity of the higher-order charges is rooted in the failure of associativity (which, in particular, invalidates the Jacobi identity), these models might be regarded as integrable in some sense (to be precise). Granting this generalization, such systems may be also conjectured to be identified by the existence of a non-zero charge H_3.

Conjecture 1 could be generalized in a natural way to include even models with long-range interactions [40], for which there seem to be no ladder operator. We would then require that if a Hamiltonian H_2 is given by a sum of two-spin interactions, there should exist a conserved three-spin charge H_3. Observe that for models with long-range interactions, the leading term in H_n is also characteristic: although the n interacting spins in the leading term are no longer adjacent, the prefactor specifying the interaction of these n spins is distinctive. But, for these models, there can exist independent non-local charges also for finite chains (see e.g. [41]), and again it could be less obvious at first sight to assert that a three-spin quantity is not a product of lower-order charges. The analysis of models with long-range interactions will be reported elsewhere.

Finally, we mention a completely different integrability test for spin chains based on the properties of the n-magnon excitations of the ferromagnetic vacuum, which has been conjectured by Haldane [42]. In spin-s chains the bound-state n-magnon dispersion branches extend over $\min(N, 2s)$ Brillouin zones and integrability manifests itself in that all branches are real (i.e. lie outside the spin wave continuum) and are continuous through the zone boundaries. This integrability criterion has been investigated for 2-magnon excitations in spin-s chains in [43, 44]. In particular, in [43] the Haldane criterion was used to obtain a relation between the parameters of a class of isotropic spin-s Hamiltonians. This relation defines then a one-parameter family of models, supposed to be integrable. But it contains only two out of the four known fundamental integrable systems for $s = \frac{3}{2}$ [14]. Neither the $su(4)$ invariant chain [29] nor the Hamiltonian H_{II} of [32] belong to it. And apart from two points (describing the Bethe ansatz integrable system [45] and the chain related to the Temperley–Lieb algebra [46–48]), the other models within the above one-parameter $s = \frac{3}{2}$ family are not fundamental integrable systems, and are very likely to be non-integrable. Thus the condition in [43] turns out to be neither a necessary nor a sufficient condition for integrability. Presumably, further constraints should follow from the analysis of n-magnon excitations with $n > 2$. A better understanding of the Haldane criterion, as well as its relation to the test proposed in this work, is clearly needed. Note that the former approach implies a choice of a particular vacuum, while the H_3 test is insensitive to the ferromagnetic or antiferromagnetic character of the model.

Appendix A. The Dolan–Grady integrability condition for self-dual systems

For self-dual quantum chains, there exists a simple sufficient condition for integrability, due to Dolan and Grady [10]. This condition actually applies to any type of self-dual system, discrete or continuous, and defined in any number of spacetime dimensions. These systems are described by a Hamiltonian of the form

$$H_2 = \alpha H + \beta \tilde{H} \tag{A.1}$$

where $\tilde{H}$ is the dual of H, with duality being defined as any non-trivial linear operator with the property $\tilde{\tilde{H}} = H$. If such Hamiltonians satisfy the relations

$$[H, [H, [H, \tilde{H}]]] = 16[H, \tilde{H}] \tag{A.2}$$

there exists a (potentially infinite, for infinite systems) family of conservation laws and these can be constructed systematically. We stress that this sufficient condition for integrability can be equivalently regarded as the requirement of the existence of a charge H_3 of a particular form. We have thus a neat situation here in which the existence of a third-order charge H_3 guarantees the existence of an infinite family of conserved charges. Unfortunately, the applicability of this condition seems rather limited. Condition (A.2) appears to apply only to the Z_n generalization of the Ising model defined in [35]. Systems satisfying (A.2) are sometimes called superintegrable and the underlying algebraic structure is the so-called Onsager algebra [36], [49].

On the other hand, many systems are self-dual in the above sense, but they do not satisfy (A.2). For example, the H_{XYZ} Hamiltonian is self-dual, as it may be put in the form

$$H_{XYZ} = \alpha H_{YZ} + \beta \bar{H}_{YZ} \tag{A.3}$$

where

$$H_{YZ} = \sum_{j\in\Lambda} [\lambda_y \sigma_j^y \sigma_{j+1}^y + \lambda_z \sigma_j^z \sigma_{j+1}^z] \tag{A.4}$$

where σ_i^a are Pauli spin matrices acting non-trivially only on the site i of the lattice Λ. The duality is defined by

$$\tilde{\sigma}_j^x = \sigma_j^y \qquad \tilde{\sigma}_j^y = \sigma_j^x \qquad \tilde{\sigma}_j^z = -\sigma_j^z \,. \tag{A.5}$$

However, condition (A.2) is not satisfied unless $\lambda_y = 0$ (in which case (A.1) reduces to the Ising model Hamiltonian) or $\lambda_z = 0$ (the XY model).

Appendix B. Integrability of the XX staggered model

The conserved charges for the XX staggered model

$$H_2 = \sum_{j\in\Lambda} (\lambda + \lambda'(-1)^j)(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) \tag{B.1}$$

can all be expressed in terms of the densities

$$e_{n,j}^{\alpha\beta} = \sigma_j^\alpha \sigma_{j+1}^z \cdots \sigma_{j+n-2}^z \sigma_{j+n-1}^\beta \tag{B.2}$$

defined for $n \geqslant 2$. In terms of these quantities, the scalar and pseudoscalar conserved charges [15] of the XX model are

$$\begin{aligned} h_n^{(+)} &= \sum_{j\in\Lambda} e_{n,j}^{xx} + e_{n,j}^{yy} \qquad && n \text{ even} \\ &= \sum_{j\in\Lambda} e_{n,j}^{xy} - e_{n,j}^{yx} && n \text{ odd} \end{aligned} \tag{B.3}$$

and

$$\begin{aligned} h_n^{(-)} &= \sum_{j\in\Lambda} e_{n,j}^{xy} - e_{n,j}^{yx} \qquad && n \text{ even} \\ &= \sum_{j\in\Lambda} e_{n,j}^{xx} + e_{n,j}^{yy} && n \text{ odd}\,. \end{aligned} \tag{B.4}$$

We also define

$$k_n^{(+)} = \sum_{j\in\Lambda} (-1)^j (e_{n,j}^{xx} + e_{n,j}^{yy}) \qquad k_n^{(-)} = \sum_{j\in\Lambda} (-1)^j (e_{n,j}^{xy} - e_{n,j}^{yx}) \,. \tag{B.5}$$

The conserved charges of the staggered XX model contain two families $H_n^{(\pm)}$. For n odd these charges coincide with the XX charges:

$$H_{2m+1}^{(+)} = h_{2m+1}^{(+)} \qquad H_{2m+1}^{(-)} = h_{2m+1}^{(-)} . \tag{B.6}$$

For n even, the conserved charges for (3.30) are

$$\begin{aligned} H_{2m}^{(+)} &= \lambda h_{2m}^{(+)} + \lambda' k_{2m}^{(+)} + \lambda h_{2m-2}^{(+)} - \lambda' k_{2m-2}^{(+)} \\ H_{2m}^{(-)} &= \lambda h_{2m}^{(-)} + \lambda' k_{2m}^{(-)} + \lambda h_{2m-2}^{(-)} - \lambda' k_{2m-2}^{(-)} . \end{aligned} \tag{B.7}$$

Mutual commutativity of the charges $H_n^{(\pm)}$ as well as their commutation with the staggered XX Hamiltonian (for $|\Lambda|$ even) can be verified directly as in [15]. Note also that the boost operator

$$B = \frac{1}{2\mathrm{i}} \sum_{j \in \Lambda} j [\lambda(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) + \lambda'(-1)^j (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y)] \tag{B.8}$$

has the ladder property: acting on (3.30) it produces the scalar part of (3.31).

Under the transformation (3.34), the alternating part of the staggered XX Hamiltonian transforms into the two-spin pseudoscalar charge $h_2^{(-)}$ of the XX model

$$\sum_{j \in \Lambda} (-1)^j (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) \to h_2^{(-)} \tag{B.9}$$

where

$$h_2^{(-)} = \sum_{j \in \Lambda} [\sigma_j^x \sigma_{j+1}^y - \sigma_j^y \sigma_{j+1}^x] . \tag{B.10}$$

This is a special case of the Dzyaloshinski–Moriya interaction [26]. Notice that transformation (3.34) can be interpreted as a duality in the sense of (A.1) if we define

$$\tilde{\sigma}_{2j+1}^x = \mathrm{i}\sigma_{2j+1}^y \qquad \tilde{\sigma}_{2j+1}^y = -\mathrm{i}\sigma_{2j+1}^x$$

(where the factor i has been introduced in order to have $\tilde{\tilde{\sigma}} = \sigma$). Then $\tilde{h}_2^{(+)} = \mathrm{i}k_2^{(-)}$ and $\tilde{h}_2^{(-)} = \mathrm{i}k_2^{(+)}$. One may then consider a general Hamiltonian

$$H = \lambda_1 h_2^{(+)} + \lambda_2 h_2^{(-)} + \lambda_3 k_2^{(-)} + \lambda_4 k_2^{(+)} \tag{B.11}$$

where $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are arbitrary constants. This Hamiltonian is integrable, as can be seen from the existence of an infinite family of conservation laws, given again (for n odd) by (B.6). Notice that (B.11) is self-dual for $\lambda_1\lambda_4 = \lambda_2\lambda_3$; however, it does not satisfy the Dolan–Grady sufficient integrability condition (A.2).

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